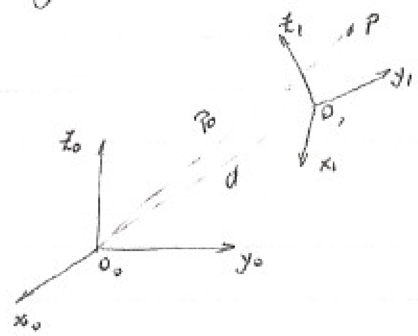
Week 6, Lecture 2 - Robotics 2008/02/12

Assume that P_1 is vector in frame (O_1, x_1, y_1, z_1) and the frame (O_1, x_1, y_1, z_1) is rotating relative to frame (O_0, x_0, y_0, z_0)



$$P_0 = d + RP_1$$
$$R = {}^0_1R$$

the velocity of P in frame $\{0\}$ is given by
$\dot{P}_0 = \dot{d} + \dot{R}P_1 + R\dot{P}_1$ let $\dot{P}_1 = 0$ (P is ~~not~~ moving with frame $\{1\}$)

$\dot{P}_0 = \dot{d} + \dot{R}P_1$, $\dot{R} = RS(\omega)$

$\dot{P}_0 = \dot{d} + S(\omega)\underbrace{RP_1}_{P_0 - d}$ $\Rightarrow \dot{P}_0 = \dot{d} + S(\omega)[P_0 - d] = \dot{d} + \omega \times (P_0 - d)$

ω is angular velocity of $\{1\}$ w.r.t. $\{0\}$

Acceleration

- we use the property $\frac{d}{dt}(a \times b) = \frac{da}{dt} \times b + a \times \frac{db}{dt}$.

let $P_0 = d + RP_1 \Rightarrow \dot{P}_0 = \dot{d} + \dot{R}P_1$ (assuming that $\dot{P}_1 = 0$ i.e. P is not moving w.r.t. frame $\{1\}$) $\Rightarrow \dot{P}_0 = \dot{d} + S(\omega)RP_1 = \dot{d} + \omega \times r$ where $r = RP_1$

acceleration of P in frame $\{0\}$ $\ddot{P}_1 = \ddot{d} + \dot{\omega} \times r + \omega \times \dot{r} = \ddot{d} + \dot{\omega} \times r + \omega \times [\dot{R}P_1]$

$\ddot{P}_1 = \ddot{d} + \dot{\omega} \times r + \omega \times [\underbrace{S(\omega)RP_1}_{\omega \times r}]$ $\Rightarrow$ $\ddot{P}_0 = \ddot{d} + \dot{\omega} \times r + \omega \times (\omega \times r)$

$\ddot{d}$: linear acceleration.

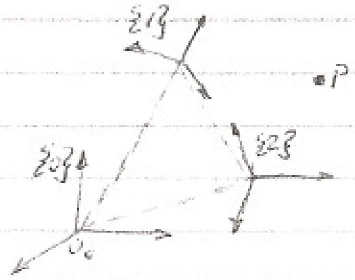
$\omega \times (\omega \times r)$: centripetal acceleration, directed towards axis of rotation and is $\perp$ to that axis

$\dot{\omega} \times r$: transverse acceleration

If P_1 is moving w.r.t $\{1\}$ then $\ddot{P}_0 = \ddot{d} + \dot{\omega} \times r + \omega \times (\omega \times r) + \underbrace{2\omega \times R\dot{P}_1}_{\text{coriolis acceleration}}$

Remember!! $\frac{dR}{dt} = S(\omega)R$

Addition of angular velocities



$P_0 = {}^0_1 d + {}^0_1 R P_1$

$P_1 = {}^1_2 d + {}^1_2 R P_2$

$P_2 = {}^0_2 d + {}^0_2 R P_2$

${}^0_2R = {}^0_1R\,{}^1_2R \to (1)$

${}^0_2R = {}^0_1 d + {}^0_1R\,{}^1_2 d \to (2)$

Differentiating: $(1) \Longrightarrow {}^0_2\dot R = {}^0_1\dot R\,{}^1_2R + {}^0_1R\,{}^1_2\dot R \to (3)$

${}^0_2\dot R = S({}^0_2\omega)\,{}^0_2R \to (4)$

${}^0_1\dot R\,{}^1_2R = S({}^0_1\omega)\,{}^0_1R\,{}^1_2R = S({}^0_1\omega)\,{}^0_2R \to (5)$

${}^0_1R\,{}^1_2\dot R = {}^0_1R\,S({}^1_2\omega)\,{}^1_2R$

${}^0_1R\,{}^1_2\dot R = {}^0_1R\,S({}^1_2\omega)\,{}^0_1R^T\,{}^0_1R\,{}^1_2R = S({}^0_1R\,{}^1_2\omega)\,{}^0_1R\,{}^1_2R = S({}^0_1R\,{}^1_2\omega)\,{}^0_2R \to (6)$

$5 \,\&\, 6 \Longrightarrow {}^0_2\dot R = S({}^0_1\omega)\,{}^0_2R + S({}^0_1R\,{}^1_2\omega)\,{}^0_2R \to (7)$

matching 4 & 7 $\Longrightarrow S({}^0_2\omega)\,{}^0_2R = S({}^0_1\omega)\,{}^0_2R + S({}^0_1R\,{}^1_2\omega)\,{}^0_2R$

$S({}^0_2\omega) = S({}^0_1\omega) + S({}^0_1R\,{}^1_2\omega) = S({}^0_1\omega + {}^0_1R\,{}^1_2\omega)$

$\Longrightarrow \quad {}^0_2\omega = {}^0_1\omega + {}^0_1R\,{}^1_2\omega$

Note: angular velocities can be added as long as they are expressed in the same frame

${}^1_2\omega$: angular velocity of frame {2} w.r.t. frame {1} expressed in frame {1}

${}^0_1R\,{}^1_2\omega$: angular velocity of frame {2} w.r.t. frame {1} expressed in {0}.

In general:

${}^0_nR = {}^0_1R\,{}^1_2R \dots {}^{n-1}_nR$

${}^0_n\dot R = S({}^0_n\omega)\,{}^0_nR$

${}^0_n\omega_n = {}^0_1\omega + {}^0_1R\,{}^1_2\omega + {}^0_2R\,{}^2_3\omega + \dots + {}^0_{n-1}R\,{}^{n-1}_n\omega$ — angular velocity of frame {n} w.r.t. {0} expressed in {0}

Manipulator Jacobian

let $${}^{0}_{n}T = \begin{bmatrix} {}^{0}_{n}R(\underline{q}) & {}^{0}_{n}d(\underline{q}) \\ 0 & 1 \end{bmatrix}$$ where $\underline{q} = [q_1, q_2, q_3 \cdots q_n]^T$.

${}^{0}_{n}T$ is the homogeneous transformation from the end-effector frame $\{n\}$ to the base frame $\{0\}$.
The objective is to find the relation between the end-effector velocities and the joint velocities.
Let ${}^{0}_{n}\omega$ be the angular velocity of the end-effector with respect to the base frame

$$S({}^{0}_{n}\omega) = {}^{0}_{n}\dot{R}\,{}^{0}_{n}R^T.$$

Let ${}^{0}_{n}V = {}^{0}_{n}\dot{d}$ denote the linear velocity of the end-effector
we're looking for equation of the form

$${}^{0}_{n}V = J_v \dot{\underline{q}}$$
$${}^{0}_{n}\omega = J_\omega \dot{\underline{q}}$$

$$\underbrace{\begin{bmatrix} {}^{0}_{n}V \\ {}^{0}_{n}\omega \end{bmatrix}}_{6\times 1} = \underbrace{J(\underline{q})}_{6\times n} \underbrace{\dot{\underline{q}}}_{n\times 1} \qquad J(\underline{q}) = \begin{bmatrix} J_v(q) \\ J_\omega(q) \end{bmatrix} \text{ manipulator Jacobian}$$

$n = \#$ of DOF's

$$J_\omega = [\rho_1 \hat{z}_0 \quad \rho_2 \hat{z}_1 \quad \cdots \quad \rho_n \hat{z}_{n-1}] \qquad \rho_i = \begin{cases} 1 & \text{if joint } i \text{ is revolute.} \\ 0 & \text{if joint } i \text{ is prismatic} \end{cases}$$

$$J_v = [\overset{3\times 1}{J_{v_1}} \quad \overset{3\times 1}{J_{v_2}} \quad \cdots \quad J_{v_n}] \qquad J_{v_i} = \hat{z}_{i-1} \times (o_n - o_{i-1}) \text{ if joint } i \text{ is revolute.}$$
$$J_{v_i} = \hat{z}_{i-1} \text{ if joint } i \text{ is prismatic.}$$

In summary
The Jacobian is given by $J = [J_1 \cdots J_n]$

$$J_i = \begin{bmatrix} z_{i-1} \times (o_n - o_{i-1}) \\ z_{i-1} \end{bmatrix} \text{ if joint } i \text{ is revolute.}$$

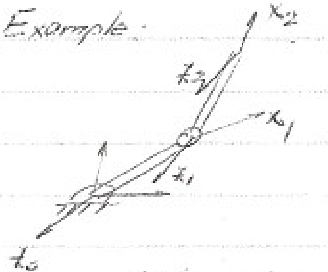
$$J_i = \begin{bmatrix} \hat{z}_{i-1} \\ 0 \end{bmatrix} \text{ if joint } i \text{ is prismatic.} \qquad i = 1, \ldots n.$$

Remark: The only quantities needed to compute the Jacobian are the unit vectors $\hat{z}_i$ expressed in the base frame $\{0\}$ and the origins $o_1, o_2 \ldots o_n$ expressed in frame $\{0\}$

* $\hat{z}_i$ is given by the 1^{st} three elements in the third column of ${}^{0}_{i}H$.
* o_i is given by the 1^{st} three elements of the fourth column of ${}^{0}_{i}H$

${}^{0}_{i}H = A_1 A_2 \dots A_n$ (D-H representation)

Example:



$z_0, z_1, z_2 \perp$ page.

$q_1 = \theta_1$

$q_2 = \theta_2$

$$J(\theta_1, \theta_2) = \begin{bmatrix} \hat{z}_0 \times (o_2 - o_0) & \hat{z}_1 \times (o_2 - o_1) \\ \hat{z}_0 & \hat{z}_1 \end{bmatrix}$$

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad o_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad z_1 = ? \quad o_1 = ? \quad o_2 = ?$$

$${}^{0}_{1}T = \begin{array}{cccc} & & \hat{z}_1 & o_1 \\ \end{array} \begin{bmatrix} c_1 & -s_1 & 0 & a_1 c_1 \\ s_1 & c_1 & 0 & a_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$${}^{0}_{2}T = A_1 A_2 = \begin{array}{cccc} & & \hat{z}_2 & o_2 \\ \end{array} \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1 c_1 + a_2 c_{12} \\ s_{12} & c_{12} & 0 & a_1 s_1 + a_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$z_0 \times (o_2 - o_0) = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} \\ 0 \end{bmatrix} \qquad z_1 \times (o_2 - o_1) = \begin{bmatrix} -a_2 s_{12} \\ a_2 c_{12} \\ 0 \end{bmatrix}$$

$$J(\theta_1, \theta_2) = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 - a_2 c_{12} & a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$